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ARROWS, SYMMETRIES AND REPRESENTATION RINGS

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To Saunders MacLane – with thanks for teaching the use of arrows and colored chalk

The aim of this paper is to present a ridiculously simple proof of a theorem on representation rings of the symmetric groups which concisely presents theorems of Frobenius, Atiyah, and Knutson. The author's intended audience is algebraic topologists and other applied mathematicians. The main technique is the use of arrows rather than summation signs [14].

Let S_n be the symmetric group on n letters and denote by $R(S_n)$ the complex representation ring of S_n. Define a graded group $R=\{R_n\}$ by setting $R_{2n+1}=0$, $R_{2n}=R(S_n)$. For $p+q=n$ let $i : S_p\times S_q\to S_n$ be the inclusion map of the subgroup which consists of permutations of the first p and the last q letters. Restriction $i^* : R(S_n)\to R(S_p)\otimes R(S_q)$ defines a coproduct $\psi : R\to R\otimes R$ and hence a product $\psi_* : R_*\otimes R_*\to R_*$ in the graded dual R_* of R. Atiyah [2] exhibited an algebra isomorphism $\Delta' : R_*\to C$, where $C=Z[y_1,\dots,y_n,\dots]$ is a graded polynomial algebra with grade $y_n=2n$. Frobenius induction $i_* : R(S_p\times S_q)\to R(S_n)$ defines a product $\varphi : R\otimes R\to R$ which makes R into a commutative and associative algebra. Frobenius [7] noticed that the algebra homomorphism $A : C\to R$ defined by $Ay_n=[n]$ (where $[n] : S_n\to U(1)$ is the trivial representation) is onto R and used it to exhibit the irreducible characters of S_n. Of course, since rank $C_{2n}=$ rank $R_{2n}=\pi(n)$, the number of partitions of n, A onto implies A is an isomorphism. Knutson [11] used notions of λ-rings to exhibit an algebra isomorphism $\Theta : R\to C$.

The topologist realizes that $C_*=H^*(BU;\ Z)$ and remembers that $H^*(BU;\ Z)$ is a Hopf algebra which is self-dual: $H_*(BU;\ Z)$ is isomorphic as a Hopf algebra to $H^*(BU;\ Z)$. The thought occurs: wouldn't it be nice if (R,φ,ψ) were a Hopf algebra and $A : C\to R$ would be an isomorphism of Hopf algebras? This would be instant explanation why R and R_* are both isomorphic to C.

The first to remark in the literature that (R,φ,ψ) is a Hopf algebra seems to be Burroughs [4]. The proof is not given, but it probably used the Mackey [13] formula for restrictions of induced representations. A proof along these lines appears in

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Section 4 of [12] which is a precursor of this paper. Here we present a proof due to A. Dress which uses the magic of arrows: to verify a given property, just write down the diagram you wish to commute. This method of proof is explained and carried out in Section 1.

The Hopf algebra $C=Z[y_1,\dots,y_n,\dots]$ with the Chern diagonal $\psi(y_n)=\sum_{i+j=n} y_i\otimes y_j$ is studied in Section 2. The group of all Hopf algebra homomorphisms $C\to C_*$ is shown to be isomorphic to the multiplicative group of power series in one variable with constant term one. The work of Newton [17] on the primitives of C is presented and shown to imply that there are exactly four Hopf algebra isomorphisms $C\to C_*$, furthermore the group Haut(C) of Hopf algebra automorphisms of C is the Klein four-group. The work of Hirzebruch [8] on multiplicative sequences fits nicely here.

The structure of R is studied in Section 3. The proof that $A : C\to R$ is an isomorphism is very easy. There is a surprise: if $S : R\to R_*$ is the Hopf algebra isomorphism given by the Schur inner product, and $B : C\to R$ any of the four Hopf algebra isomorphisms, then $B_*SB=H$ is independent of B. We show how the Hopf structure determines the algebra structure of $R(S_n)$ under tensor product of representations as multiplication. The irreducible representations of S_n are determined and the corresponding elements $b_E\in C_{2n}$ are shown to arise by applying the Gram–Schmidt orthogonalization process to a standard monomial basis.

1. The language of equivariant K-theory

Let G be a group, X a G-space and $K_G(X)$ the Grothendieck group of complex G vector bundles over X (see [3, 19]). In our applications G will be a finite group, X a finite set, and G and X will have the discrete topology. For example, if $X=*$ is a point, then $K_G(*)=R(G)$, the complex representation ring of G. More generally, if H is a subgroup of G, then $K_G(G/H)\cong R(H)$, the isomorphism induced by the function which assigns to a representation α of H the G-vector bundle $G\times_H\alpha$ over G/H.

If $f : Y\to X$ is a G-map, let $f^! : K_G(X)\to K_G(Y)$ be the homomorphism which to a G-vector bundle β associates the pullback $f^!\beta$. For example, if $\pi : G/H\to *$ is the collapsing map, then $\pi^! : K_G(*)\to K_G(G/H)$ corresponds to the restriction $i^* : R(G)\to R(H)$, since if V is a representation of G, then $G\times_H i^*V$ is isomorphic to $G/H\times V$ as G-vector bundle (the map $[g,v]\to(gH,gv)$ does it).

If $\pi : Y\to X$ is a finite covering we define $\pi_! : K_G(Y)\to K_G(X)$ by associating to a G-vector bundle β over Y the G-vector bundle $\pi_!\beta$ over X with fiber

$$(\pi_!\beta)_x=\bigoplus_{y\in\pi^{-1}(x)}\beta_y.$$

For example, if $\pi : G/H\to *$ is the collapsing map and W is a representation of H, then $\pi_!(G\times_H W)=i_*W=C[G]\times_{C[H]}W$, the representation W induced up to G.

Now suppose $\pi : Y \to X$ is a G-map which is a finite covering and $f : X' \to X$ is a G-map. We have the pullback diagram of G-maps

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "\pi'"] & Y \arrow[d, "\pi"] \\ X' \arrow[r, "f"] & X \end{tikzcd}$$

where π' is a finite covering. Mackey's theorem [13] in this context can be restated simply as $f^!\pi_! = \pi'_! f'^!$. The constructions $(\ )^!$ and $(\ )_!$ are related by a further identity: $\pi_!(\pi^!(a)b) = a\pi_!(b)$, where the product in $K_G(\ )$ is induced by tensor product of vector bundles. The triple $(K_G(\ ), (\ )^!, (\ )_!)$ is an example of what Dress [5] calls a Mackey functor.

If X is a finite G-set, it is easy to determine $K_G(X)$. First decompose X into orbits

$$X \cong \coprod_i G/H_i,$$

then evaluate:

$$K_G(X) \cong \bigoplus_i R(H_i).$$

Here is an example: let $A = \{1, 2\}$ be a two element set, $X = A^n$ the n-fold Cartesian product of A with itself. Let $G = S_n$ be the symmetric group on n letters and let it act on X by permuting coordinates. There are precisely $n+1$ orbits of S_n in X, namely the orbits of $x_i = (1, \ldots, 1, 2, \ldots, 2)$ where the first i coordinates of x_i are 1, the last $n-i$ are 2. The isotropy group of x_i in S_n is $S_i \times S_{n-i}$, so we have

$$X = A^n \cong \coprod_i S_n/S_i \times S_{n-i}$$

and

$$K_{S_n}(A^n) \cong \bigoplus_i R(S_i) \otimes R(S_{n-i}),$$

where we have used $R(G \times H) \cong R(G) \otimes R(H)$.

Consider the collapsing map $p : A \to *$. For each natural number n we have the collapsing map $p^n : A^n \to *^n = *$, and the maps

$$(p^n)^! : R(S_n) \to K_{S_n}(A^n), \qquad (p^n)_! : K_{S_n}(A^n) \to R(S_n).$$

Let $i : S_p \times S_q \to S_n$ for $p + q = n$ be the inclusion. In terms of the decomposition of $K_{S_n}(A^n)$ we have just given, if $a \in R(S_n)$ then the $S_p \times S_q$-component of $(p^n)^! a = i^* a$, the restriction of a to $S_p \times S_q$. Similarly, if $b \in R(S_p \times S_q)$, then $(p^n)_! b = i_* b$, the induction of b up to S_n.

We define a graded abelian group R by setting $R_{2n} = R(S_n)$, $R_{2n+1} = 0$. Define a product $\varphi : R \otimes R \to R$ and a coproduct $\psi : R \to R \otimes R$ by setting

$$\varphi = (p^n)_! : (R \otimes R)_{2n} = K_{S_n}(A^n) \to R_{2n},$$

$$\psi = (p^n)^! : R_{2n} = R(S_n) \to K_{S_n}(A^n).$$

Define a unit $\eta : Z \to R$ by $\eta(1) = [0]$, the trivial representation $S_0 \to U(1)$, and a counit $\varepsilon : R \to Z$ defined by $\varepsilon[0] = 1$, $\varepsilon(x) = 0$ if grade $x > 0$.

Theorem A. *$(R, \varphi, \psi, \eta, \varepsilon)$ is a bicommutative, biassociative connected Hopf algebra.*

Proof. The idea is very simple: just write down the diagram wished to commute. For example

$$\begin{tikzcd}
1 \arrow[dr] & 2 \arrow[dl, "\tau"'] \arrow[r, "p"] & * \arrow[d, "1"] \\
1 & 2 \arrow[r, "p"] & *
\end{tikzcd}$$

is a pullback diagram. For each n we have the pullback diagram

$$\begin{tikzcd}
A^n \arrow[r, "p^n"] \arrow[d, "\tau^n"'] & * \arrow[d, "1"] \\
A^n \arrow[r, "p^n"] & *
\end{tikzcd}$$

with $(\tau^n)^! = T : (R \otimes R)_{2n} \to (R \otimes R)_{2n}$ the twist map. We now obtain

$$T\psi = (\tau^n)^!(p^n)^! = (p^n\tau^n)^! = (p^n)^! = \psi,$$

and

$$\varphi T = (p^n)_!(\tau^n)^! = 1^!(p^n)_! = (p^n)_! = \varphi,$$

so ψ and φ are commutative.

To prove associativity, inspect the commutative diagram

$$\begin{tikzcd}
D \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "p"] \\
A \arrow[r, "p"] & *
\end{tikzcd}$$

where $D = \{1, 2, 3\}$ and

$$\begin{tikzcd}[row sep=small]
1 \arrow[rrr, "f"] \arrow[dddd, "g"'] & & & 1 \\
& 2 \arrow[urr] \arrow[ddd] & & \\
& & 3 \arrow[r] \arrow[ddl] & 2 \arrow[dd, "p"] \\
& & & \\
1 & 2 \arrow[rr, "p"] & & *
\end{tikzcd}$$

For each n we have $Ks_n(D^n) = (R \otimes R \otimes R)_{2n}$

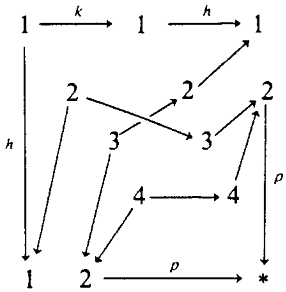
$$\varphi \circ (\varphi \otimes 1) = (p^n)_!(f^n)_! = (p^n f^n)_! = (p^n g^n)_! = (p^n)_!(g^n)_! = \varphi \circ (1 \otimes \varphi),$$

$$(\psi \otimes 1)\psi = (f^n)^!(p^n)^! = (p^n f^n)^! = (p^n g^n)^! = (g^n)^!(p^n)^! = (1 \otimes \psi)\psi.$$

It remains to prove that ψ is a homomorphism of algebras. We consider the set $B = \{1, 2, 3, 4\}$ and the commutative diagram

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d] & B \arrow[r, "h"] & A \arrow[d] \\ A \arrow[rr, "p"] & & * \end{tikzcd}$$

given by



and notice that in grade $2n$

$$\psi\varphi = (p^n)^!(p^n)_! = (h^n)_!(kh)^! = (h^n)_!(k^n)^!(h^n)^!$$
$$= (\varphi \otimes \varphi)(1 \otimes T \otimes 1)(\psi \otimes \psi),$$

which verifies the Hopf condition and proves Theorem A.

We shall need two more bits of information about the graded algebra R. First, $R(S_n)$ is free abelian of rank $\pi(n) =$ number of partitions of n. This follows since rank $R(S_n) =$ number of conjugacy classes of S_n, and conjugacy classes are indexed by partitions of n (indicating the cycle structure). Second, if we let $[n] : S_n \to U(1)$ be the trivial representation, then

$$\psi[n] = \sum_{i+j=n} [i] \otimes [j].$$

Our next aim is to prove

Theorem. $R = Z[[1], [2], \ldots, [n], \ldots]$.

In order to prove this we investigate the abstract Hopf algebra $C = Z[y_1, \ldots, y_n, \ldots]$ where grade $y_n = 2n$ and $\psi(y_n) = \sum_{i+j=n} y_i \otimes y_j$. Our study of C will be via its universal properties (see Moore [16]).

2. The classical Hopf algebra C

Let $Z[t]$ be the graded polynomial algebra on an indeterminate t of grade 2. Let $\Gamma = Z[t]_*$ be its graded dual with generators y_n defined by $\langle y_n, t^n \rangle = 1$. We let $C = Z[y_1, \ldots, y_n, \ldots]$, $y_0 = 1$, and define the Hopf algebra structure on C by making the inclusion $i : \Gamma \to C$ into a map of coalgebras.

The map i has the following universal property: if A is a graded associative algebra and $\varphi : \Gamma \to A$ is a homomorphism of graded groups then there exists a unique homomorphism of graded algebras $\Phi : C \to A$ such that $\Phi i = \varphi$, that is, the commutativity of the diagram

$$\begin{tikzcd} C \arrow[d, "i"'] \arrow[dr, "\Phi"] & \\ \Gamma \arrow[r, "\varphi"] & A \end{tikzcd}$$

characterizes Φ. If A is a Hopf algebra and φ is a map of coalgebras then Φ is a map of Hopf algebras.

Taking graded duals over Z we obtain a map of algebras $i_* : C_* \to Z[t]$ which has the following property: if B is a graded coalgebra with each B_n free abelian of finite rank, then given a homomorphism of graded groups $\theta : B \to Z[t]$ there exists a unique homomorphism of coalgebras $\Theta : B \to C_*$ such that $i_* \Theta = \theta$. That is, the commutative diagram

$$\begin{tikzcd} & C_* \arrow[d, "i_*"] \\ B \arrow[ur, "\Theta"] \arrow[r, "\theta"] & Z[t] \end{tikzcd}$$

characterizes Θ. If B is a Hopf algebra and θ is a map of algebras then Θ is a map of Hopf algebras.

We will use these universal properties to study Hopf algebra homomorphisms $\Sigma : C \to C_*$. Let $\mathrm{Hhom}(C, C_*)$ be the set of all such Σ. We define addition by letting $\Sigma_1 + \Sigma_2 = \psi_*(\Sigma_1 \otimes \Sigma_2)\psi$. Let us denote by $\mathrm{Ghom}(\Gamma, Z[t])$ the set of all grade-preserving homomorphisms $s : \Gamma \to Z[t]$ with $s(y_0) = 1$. We define addition in this set by

$$(s_1 + s_2)(y_n) = \sum_{i+j=n} s_1(y_i) s_2(y_j).$$

Theorem B. *The function*

$$R : \mathrm{Hhom}(C, C_*) \to \mathrm{Ghom}(\Gamma, Z[t])$$

which to Σ associates $i_* \Sigma i$ is an isomorphism. If $\Sigma : C \to C_*$, then $\Sigma_* = \Sigma$ under the canonical identification of C_{**} with C.

Proof. Since i_* is a homomorphism of algebras we have $R(\Sigma_1+\Sigma_2)=R(\Sigma_1)+R(\Sigma_2)$. We define a function $T:\mathrm{Ghom}(\Gamma, Z[t])\to \mathrm{Hhom}(C, C_*)$ as follows. If $s:\Gamma\to Z[t]$ is a homomorphism of graded groups with $s(y_0)=1$, define $\sigma: C\to Z[t]$ to be the algebra homomorphism making

$$
\begin{tikzcd}
C \arrow[dr, "\sigma"] & \\
\Gamma \arrow[u, "i"] \arrow[r, "s"'] & Z[t]
\end{tikzcd}
$$

commute. Taking graded duals, the diagram

$$
\begin{tikzcd}
 & C_* \arrow[d, "i_*"] \\
\Gamma \arrow[ur, "\sigma_*"] \arrow[r, "s_*=s"'] & Z[t]
\end{tikzcd}
$$

commutes and σ_* is a map of coalgebras, so there exists a Hopf algebra map $T(s): C\to C_*$ making the following diagram commute:

$$
\begin{tikzcd}
C \arrow[r, "T(s)"] & C_* \arrow[d, "i_*"] \\
\Gamma \arrow[u, "i"] \arrow[ur, "\sigma_*"] \arrow[r, "s"'] & Z[t]
\end{tikzcd}
$$

We claim: $RT=$ identity of $\mathrm{Ghom}(\Gamma, Z[t]_*)$. This is easy:

$$RT(s)=i_*T(s)i=i_*\sigma_*=(\sigma i)_*=s_*=s.$$

To prove that $TR(\Sigma)=\Sigma$ is equally easy: it is sufficient to show that $TR(\Sigma)i=\Sigma i$, but these are coalgebra maps, so this is true only if $i_*TR(\Sigma)i=i_*\Sigma i$, that is, $RTR(\Sigma)=R(\Sigma)$. But we have already shown that $RT=$ identity, so we are done: $TR=$ identity. Finally, $R(\Sigma)=R(\Sigma_*)$, so $\Sigma=\Sigma_*$.

The thing to remember in the proof is that we are aiming to show that Σ is characterized as the Hopf algebra homomorphism which makes the following diagram commute:

$$
\begin{tikzcd}
C \arrow[r, "\Sigma"] \arrow[dr, "\sigma"] & C_* \arrow[d, "i_*"] \\
\Gamma \arrow[u, "i"] \arrow[ur, "\sigma_*"] \arrow[r, "s"'] & Z[t]
\end{tikzcd}
$$

where $s=i_*\Sigma i$.

Corollary 1. *$\mathrm{Hhom}(C, C_*)$ is isomorphic to the multiplicative group of power series over Z with constant term 1 under the isomorphism which to $a : C \to C_*$ assigns the power series $\sum_n a(y_n)(y_n)t^n$.*

Proof. The map $i_* : C_* \to Z[t]$ is given by $i_*(f) = f(y_n)t^n$ if $f \in C_{*2n}$. The group $\mathrm{Ghom}(\Gamma, Z[t])$ is isomorphic to the multiplicative group of power series over Z with constant term 1 via the function which to $s : \Gamma \to Z[t]$ assigns the power series $\sum_n s(y_n)t^n$.

We will now study which of the Hopf algebra homomorphisms are monomorphisms. The easiest way of checking that $\Sigma : C \to C_*$ is a monomorphism is to check that it is a monomorphism on the *primitives* of C: these are x of positive degree with $\psi(x) = x \otimes 1 + 1 \otimes x$. I. Newton studied the primitives of C and obtained the following fundamental result.

Theorem C (Newton). *If F is a commutative ring with unit (concentrated in degree zero), then the primitives of $F \otimes C_{2n}$ are all of the form $f \otimes p_n$, where $f \in F$ and p_n satisfies the recursion relation*

$$p_n + y_1 p_{n-1} + \cdots + y_{n-1} p_1 - n y_n = 0.$$

Proof. Let grade $x_i = 2$ and define an algebra homomorphism $f_N : C \to Z[x_1, \ldots, x_N]$ by setting $f_N(y_k) = j_k(x_1, \ldots, x_N)$, the kth elementary symmetric function of $x_1, \ldots, x_N$. In gradings $\leq 2N$, f_N is a monomorphism with image f_N being the symmetric polynomials. Given a partition $\pi = (k_1, \ldots, k_r)$ of $k = k_1 + \cdots + k_r$, $r \leq N$, we define $s(\pi)$ to be the sum of the monomials in the orbit of $x_1^{k_1} \cdots x_r^{k_r}$ under S_N. For example, $\sigma_k = s(1, \ldots, 1)$, $\pi = (1$ k-times). The diagonal $\psi : C \to C \otimes C$ corresponds to

$$\psi s(\pi) = \sum_{(\pi_1, \pi_2) = \pi} s(\pi_1) \otimes s(\pi_2),$$

the sum ranging over subpartitions π_1 of π. The only partition of k having no proper subpartitions is (k). Let $s_k = s((k))$. Newton noticed that s_k satisfy the recursion relation

$$s_n - s_{n-1} y_1 + \cdots + (-1)^{n-1} s_1 y_{n-1} + (-1)^n n y_n = 0.$$

Indeed, if we let $a = \prod_{i=1}^n (x - x_i) = x^n - y_1 x^{n-1} + \cdots + (-1)^n y_n$, then using $N = n$, $s_k = x_1^k + \cdots + x_n^k$ we obtain the relation from $a(x_1) + \cdots + a(x_n) = 0$. Now letting $p_n = (-1)^{n+1} s_n$ we obtain Theorem C.

Scholium. Isaac Newton worked out the recursion relation for $n \leq 8$ around 1665 [18, vol. I, p. 519]. The formulae for s_2, s_3 and s_4 had been published by Albert Girard in 1629, but there are no indications that Newton was aware of this. Newton gives the recursion formula in [17, p. 166] (see [18, vol. III, p. 361]) in the sense that he writes out the formula for low values of n and says "et sic in infinitum observata serie progressionis." The first published proof of the recursion formula seems to be in Colin Maclaurin's posthumous work [15, part II, chapter XII].

Lemma 2. *Let $\mathscr{H} : C \to Z[t]$ be the homomorphism of algebras defined by $\mathscr{H}(y_n) = t^n$ for all n. Then we have $\mathscr{H}(p_n) = t^n$ as well.*

Proof. $\mathscr{H}(p_n) = t^n$ satisfies the Newton recursion relation.

Corollary 3. *The Hopf algebra homomorphism $H : C \to C_*$ defined by $i_* H = \mathscr{H}$ of Lemma 2 is an isomorphism.*

Proof. Since rank $C_{2n} =$ rank $C_{*2n} = \pi(n)$, the number of partitions of n, we only have to show that $1 \otimes H : F \otimes C \to F \otimes C_*$ is a monomorphism for all quotient rings F of Z. We have the commutative diagram

$$\begin{tikzcd} F \otimes C \arrow[r, "1 \otimes H"] \arrow[dr, "1 \otimes \mathscr{H}"'] & F \otimes C_* \arrow[d, "1 \otimes i_*"] \\ & F \otimes Z[t] = \mathrm{F}[t] \end{tikzcd}$$

$(1 \otimes \mathscr{H})(1 \otimes p_n) = t^n$, so $(1 \otimes H)(1 \otimes p_n) \neq 0$, and $1 \otimes H$ is a monomorphism.

Lemma 4. *Let $\mathscr{K}, \mathscr{L}, \mathscr{M} : C \to Z[t]$ be defined by power series $1 + t$, $1 - t$, $(1 + t)^{-1}$, respectively, then $\mathscr{K}(p_n) = (-1)^{n+1} t^n$, $\mathscr{L}(p_n) = -t^n$, $\mathscr{M}(p_n) = (-1)^n t^n$.*

Corollary 5. *$K, L, M : C \to C_*$ are Hopf algebra isomorphisms.*

Our next task is to show that H, K, L, M are the only Hopf algebra isomorphisms.

Lemma 6. *If $\varphi : C \to C$ is a Hopf algebra automorphism and $\varphi p_1 = p_1$, $\varphi p_2 = p_2$, then φ is the identity.*

Proof. We shall show that $\varphi y_n = y_n$ for all n. This is true for $n = 1, 2$ by hypothesis. We assume $n > 2$ and $\varphi | C_k$ for $k < 2n$ is the identity. This means that $\varphi y_n - y_n$ is primitive, so $\varphi y_n = y_n + a p_n$, $a \in Z$. Since φ is an automorphism, $\varphi y_n = \pm y_n$ modulo decomposables. Thus looking at the coefficient of y_n we obtain $\pm 1 = 1 + an$ so either $0 = an$ (so $a = 0$ and we are done), or $-2 = an$ (which is impossible since $n > 2$). This does the inductive step, hence proves the lemma.

We shall use the scalar product notation $\langle\ ,\ \rangle : C_* \times C \to Z$ for the evaluation map.

Lemma 7. *If $N: C \to C_*$ is a Hopf algebra homomorphism, then $N(p_n) = \langle Np_n, y_n \rangle H(p_n)$.*

Proof. If $x \in C_{*2n}$, $i_* x = \langle x, y_n \rangle t^n$. Thus $\langle Hp_n, y_n \rangle t^n = i_* Hp_n = \mathscr{H}(p_n) = t^n$, hence $\langle Hp_n, y_n \rangle = 1$. Now $Np_n = a_n Hp_n$, thus $\langle Np_n, y_n \rangle = a_n \langle Hp_n, y_n \rangle = a_n$.

This allows us to complete the following table (just use Lemma 4).

Table
Hopf algebra automorphisms of C with $H^{-1}Np_n = \varepsilon_n p_n$

N	ε_1	ε_2
H	1	1
K	1	-1
L	-1	-1
M	-1	1

Corollary 8. *The group* Haut(C) *is the Klein four-group.*

Corollary 9. *If $\lambda \in$ Haut(C), then $\lambda_* H \lambda = H$.*

Proof. Let $\lambda p_n = \varepsilon p_n$, where $\varepsilon^2 = 1$. Then Theorem C tells us that $\lambda y_n = \varepsilon y_n$ modulo decomposables. Now $\langle Hp_n, \ \rangle$ kills decomposables, and we have $\langle \lambda_* H \lambda p_n, y_n \rangle = \langle H\lambda p_n, \lambda y_n \rangle = \langle \varepsilon H p_n, \varepsilon y_n \rangle = \varepsilon^2 \langle Hp_n, y_n \rangle = 1$, so $\lambda_* H \lambda = H$, by Theorem B.

The topologist knows C as $H_*(BU; Z)$ with y_n coming from $H_{2n}(CP^\infty; Z)$ (see Adams [1] for analogues in generalized homology theories). The Hopf algebra automorphism $H^{-1}L$ is $H_*(-1)$, where $-1 : BU \to BU$ is the stable normal bundle. Equivalently, $H^{-1}L = -I$ in the group Hhom(C, C) with sum defined by $f + g = \varphi(f \otimes g)\psi$, where $I : C \to C$ is the identity map. That is, $H^{-1}L$ is the conjugation in the connected Hopf algebra C. The automorphism $H^{-1}M = H_*(Bc)$, where $c : U \to U$ is complex conjugation and $Bc : BU \to BU$ is the induced map on classifying spaces. The automorphism $H^{-1}K = H_*(-1)H_*(Bc)$ tends to be neglected. We shall see that nature loves $H^{-1}K$. Here is the first instance.

It is usual to think of $C_* = H^*(BU; Z)$ as a polynomial algebra in the Chern classes c_n, where $\langle c_n, y_1^n \rangle = 1$, $\langle c_n, y^E \rangle = 0$ if $E \neq (n, 0, \dots, 0)$. Let $N : C \to C_*$ be the Hopf algebra isomorphism $Ny_n = c_n$. Then $Ny_1 = Hy_1$, but $Ny_2 \neq Hy_2$, for $\langle Ny_2, y_2 \rangle = 0$, $\langle Hy_2, y_2 \rangle = 1$. Now $N = H\lambda$, where $\lambda \in$ Haut(C). Since $\lambda p_1 = p_1$ and $\lambda \neq I$, $\lambda = H^{-1}K$ and $N = K$. The corresponding homomorphism $n = i_* N i : Z[t]_* \to Z[t]$ is given by $n(y_1) = t$, $n(y_k) = 0$ if $k > 0$. This together with Theorem B reinterprets the work of Hirzebruch [8] on multiplicative sequences, that is, on Hopf algebra homomorphisms $a : C \to C$. The characteristic power series associated to a is $\sum_k pa(y_k)$, where $p : C \to Z[t]$ is the algebra homomorphism $p(y_1) = t$, $p(y_k) = 0$ for $k > 1$.

3. The structure of R

If V and W are representations of a finite group G, we let $(V, W) = \dim_C \mathrm{Hom}_{C[G]}(V, W)$. The function (,) induces an inner product (called the Schur inner product) on the representation ring $R(G)$. Schur's Lemma asserts that the irreducible representations form an orthonormal basis of $R(G)$. If $i : H \subset G$ is the inclusion of a subgroup, then Frobenius induction $i_* : R(H) \to R(G)$ is adjoint to restriction $i^* : R(G) \to R(H)$ under the Schur inner products: $(i_* a, b)_G = (a, i^* b)_H$ for all $a \in R(H)$, $b \in R(G)$.

As before, we let $(R, \varphi, \psi, \eta, \varepsilon)$ be the graded connected Hopf algebra $R_{2n} = R(S_n)$, $R_{2n+1} = 0$, $\varphi =$ induction, $\psi =$ restriction corresponding to the inclusions $S_p \times S_q \to S_{p+q}$. Let $[n] \in R_{2n} = R(S_n)$ be the trivial representation $[n] : S_n \to U(1)$. The Schur inner product (,) defines a Hopf algebra isomorphism

$$S : (R, \varphi, \psi, \eta, \varepsilon) \to (R_*, \psi_*, \varphi_*, \eta_*, \varepsilon_*).$$

Theorem D. *The homomorphism of algebras $A : C \to R$ defined by $Ay_n = [n]$ is an isomorphism of Hopf algebras such that $A_* SA = H$.*

Proof. If we can prove that the diagram

$$\begin{tikzcd} C \arrow[r, "A"] \arrow[d, "H"'] & R \arrow[d, "S"] \\ C_* \arrow[r, "A_*"] & R_* \end{tikzcd}$$

commutes, then Theorem D will follow. We notice that R_{2n} and C_{2n} are free abelian of the same rank ($= \pi(n) =$ number of partitions of n). Thus A is an isomorphism if and only if for every quotient ring F of Z the function $1 \otimes A : F \otimes C \to F \otimes R$ is a monomorphism. But we have already seen that $1 \otimes H$ is a monomorphism for all F, so $1 \otimes A$ is a monomorphism.

To prove $A_* SA = H$ we have only to show $R(A_* SA) = R(H) = \mathscr{H}$ (Theorem B), that is, $\langle A_* SAy_n, y_n \rangle = \langle Hy_n, y_n \rangle = 1$ for all n.

Now $\langle A_* SAy_n, y_n \rangle = \langle SAy_n, Ay_n \rangle = \langle S[n], [n] \rangle = ([n], [n]) = 1$, and we are done.

Theorem D says that if $a, b \in C_{2n}$, then $(Aa, Ab) = \langle Ha, b \rangle$. There are four Hopf algebra isomorphisms $B : C \to C_*$, and we may suspect that $B_* SB$ will realize the four Hopf algebra isomorphisms $C \to C_*$. There is a surprise.

Corollary 10. *If $B : C \to R$ is a Hopf algebra isomorphism, then for all $a, b \in C_{2n}$ we have $(Ba, Bb) = \langle Ha, b \rangle$, that is $B_* SB = H$.*

Proof. Let $\lambda = A^{-1}B : C \to C$ or $B = A\lambda$, $\lambda \in \mathrm{Haut}(C)$. Then $B_* SB = \lambda_* A_* SA\lambda = \lambda_* H\lambda = H$, the last equality by Corollary 9.

Scholium. Atiyah [2] constructed an algebra isomorphism $\Delta' : R_* \to C_*$ which is described by $\Delta'(([n],\ )) = Hy_n$, that is, $E\Delta'_* = H$, so $\Delta'_* = A$. Knutson's [11] isomorphism $\Theta : R \to C$ is given by $\Theta = \alpha A^{-1}$, where $\alpha = H^{-1}K$.

Each $R_{2n} = R(S_n)$ is an algebra in its own right under the product $\times$ (induced by tensor product of representations of S_n). The corresponding product $\times$ on C_{2n} is completely determined by the Hopf algebra structure.

Corollary 11. *If $c \in C_{2n}$, then $y_n \times c = c$,*

$$(ab) \times c = \sum (a \times c')(b \times c'')$$

where $\psi(c) = \sum c' \otimes c''$ and in the first sum grade $a =$ grade c'.

Proof. If $i : S_p \times S_q \to S_n$ is the standard inclusion, $(p+q=n)$, we then have $(ab) \times u = i_*(a \otimes b) \times u = i_*(a \otimes b \times i^*u)$.

Let $\pi : S_n \to U(n)$ be the permutation representation and let $d_n = \det \pi$. Notice that $\psi(d_n) = \sum_{i+j=n} d_i \otimes d_j$. Consider the map $D : R \to R$ given by $D(a) = a \times d_n$ for $a \in R(S_n)$.

Lemma 12. *D is an automorphism of Hopf algebras and $DA = A\alpha$, where $\alpha = H^{-1}K$.*

Proof. Let $a \in R(S_p)$, $b \in R(S_q)$ then

$$D(i_*(a \otimes b)) = i_*(a \otimes b) \times d_{p+q} = i_*(a \otimes b \times i^*d_{p+q})$$
$$= i_*((a \times d_p) \otimes (b \times d_q)) = D(a)D(b).$$

Also, if $c \in R(S_{p+q})$, then $(i^*c) \times d_p \otimes d_q = i^*(c \times d_{p+q})$, so $(D \otimes D)\psi = \psi D$. Since $DD = 1$, D is a Hopf algebra automorphism. We have

$$\begin{tikzcd} C \arrow[r, "A"] \arrow[d, "\lambda"] & R \arrow[d, "D"] \\ C \arrow[r, "A"] & R \end{tikzcd}$$

where $\lambda \in \mathrm{Haut}(C)$. Since $DAy_1 = [1] \times d_1 = d_1 = [1]$, $\lambda y_1 = y_1$, so $\lambda = I$ or $\alpha = H^{-1}K$. Since $DAy_2 = [2] \times d_2 = d_2 \neq [2]$, $\lambda \neq I$, so $\lambda = \alpha$.

Corollary 13. *If E and F are exponent sequences, then $(Ay^E, DAy^F) = \langle c^E, y^F \rangle$.*

Proof. $(Ay^E, DAy^F) = (Ay^E, A\alpha y^F) = \langle Hy^E, \alpha y^F \rangle = \langle \alpha_* Hy^E, y^F \rangle$. Since $\alpha_* H\alpha = H$ and $\alpha^2 = 1$, $\alpha_* H = H\alpha = K$. Finally, $Ky^E = c^E$, and we are done.

If E and F are exponent sequences, we shall say that $E > F$ if the first nonzero entry from the right in $E-F=(e_1-f_1, e_2-f_2, \dots, e_s-f_s, \dots)$ is positive. An exponent sequence $E=(e_1,\dots,e_s)$ determines a *partition* $\pi(E)$ of $n=e_1+\cdots+se_s$ in which i occurs precisely e_i times, and a *conjugate partition* $\pi'(E)=\{e_1+\cdots+e_s, e_2+\cdots+e_s,\dots,e_s\}$. We shall write E' for the exponent sequence with $\pi(E')=\pi'(E)$. For example, if $E=(2,1,2)$, then $\pi(E)=\{1,1,2,3,3\}$, $\pi'(E)=\{5,3,2\}$, and $E'=(0,1,1,0,1)$. The following is the key to Van de Velde's proof [20] that $K: C\to C_*$ is an isomorphism.

Lemma 14. *If* $E>F$, *then* $\langle c^E, y^{F'}\rangle=0$, *but* $\langle c^E, y^{E'}\rangle=1$.

Proof. Recall that c_n takes the value 1 on y_1^n and kills the other monomials. Given an exponent sequence F, we are interested in the terms of the form $y_1^{a_1}\otimes\cdots\otimes y_1^{a_k}$ in $\psi^k(y^F)$, where $0<a_1\le a_2\le\cdots\le a_k$. If $y^F\in C_{2n}$ then $\{a_1,\dots,a_k\}$ is a partition of n. The largest such partition that we can obtain for $F=(f_1,\dots,f_s)$ is

$$y_1^{f_1+\cdots+f_s}\otimes y_1^{f_2+\cdots+f_s}\otimes\cdots\otimes y_1^{f_s},$$

that is, the exponents form the partition $\pi'(F)=\pi(F')$. In particular, if $E>F'$, then $\langle c^E, y^F\rangle=0$. If $E=F'$ (that is $E'=F$) the top term above has coefficient 1, so $\langle c^E, y^{E'}\rangle=1$.

Let $\{\varrho_1,\dots,\varrho_m\}$ be the distinct up to isomorphism irreducible complex representations of a finite group G. If σ and τ are representations of G, then $\sigma=\sum s_i\varrho_i$, $\tau=\sum t_i\varrho_i$ where $s_i=(\sigma,\varrho_i)$ and $t_i=(\tau,\varrho_i)$ are natural numbers. We thus have $(\sigma,\tau)=\sum s_it_i$. This means, for example, that if $(\sigma,\tau)=1$ then there exists exactly one i with $s_i=t_i=1$ and for $j\ne i$, $s_jt_j=0$. Again if $(\sigma,\tau)=0$ and $s_j\ne0$, then $t_j=0$.

We interpret this in R. Given the exponent sequence $E=(e_1,\dots,e_s)$ we consider the representations Ay^E and $DAy^{E'}$. We have $(Ay^E, DAy^{E'})=\langle c^E, y^{E'}\rangle=1$, so there exists a unique irreducible representation $[\pi(E)]$ of S_n occurring with coefficient 1 in Ay^E and $DAy^{E'}$.

Corollary 15. $([\pi(F)], Ay^E)=0$ *for* $E>F$.

Proof. Since $([\pi(F)], DAy^{F'})=1$ we have only to show $(DAy^{F'}, Ay^E)=0$. But $(Ay^E, DAy^{F'})=\langle c^E, y^{F'}\rangle=0$ if $E>F$, so we are done.

Scholium. This shows that if $\pi_1\ne\pi_2$ are distinct partitions of n then $[\pi_1]\ne[\pi_2]$. This means that all irreducible representations of S_n arise this way.

Let b_E be defined by $Ab_E=[\pi(E)]$.

Corollary 16. *The elements* b_E *are described by the recursion relation*

$$b_E=y^E-\sum_{F>E}\langle Hb_F, y^E\rangle b_F$$

and $b_{\Delta_n}=y_n$, *where* $\Delta_n=(0,\dots,0,1)$, $e_n=1$, $e_i=0$ *for* $i\ne n$.

Proof. $y^E = \langle Hb_E, y^E \rangle b_E + \sum_{F \neq E} \langle Hb_F, y^E \rangle b_F$. Now

$$\langle Hb_E, y^E \rangle = (Ab_E, Ay^E) = ([\pi(E)], Ay^E) = 1,$$

$$\langle Hb_F, y^E \rangle = ([\pi(F)], Ay^E) = 0 \quad \text{if } E > F.$$

Scholium. Another way of saying this is: the irreducible representations $[\pi]$ of S_n are obtained by applying the Gram–Schmidt process to the representations Ay^E which are ordered by the (decreasing) lexicographic ordering from the right on exponent sequences. This approach should be compared to Fox [6] and Kerber [10].

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